3

Graph Theoretic Formulation of Network Equations

In Chapter 2 we introduced the simplest methods for formulating network equations. The two methods considered are limited in the types of ideal elements that can be included. More advanced formulation methods without restrictions on the type of circuit elements will be given in Chapter 4. They require at least an elementary knowledge of graph theory.

Graph theory is a broad, rapidly growing mathematical discipline with many applications in engineering and computer science. Our objective in this book is to develop computer methods for the formulation, analysis, and design of electrical networks. For this reason, we will restrict the presentation of graph concepts to the material suitable for this application. The motivation for the use of a graph is evident if we consider any network with two terminal components and its nodal formulation. Write the matrix equation; then replace each passive element by some other passive element of a different type and write the equation again. The entries of the matrices will differ but the structures will be the same; the structure does not depend on elements, only on the interconnection. It is possible to deduce some general properties of networks by considering only the interconnections, and graph theory is the tool for doing so.

In Section 3.1 graph concepts are introduced, and the KCL and KVL are stated. Section 3.2 introduces the incidence matrix and shows how Kirchhoff laws can be written in terms of it. Sections 3.1, 3.2, and 3.7 are necessary for a proper understanding of the material presented in this book. The remaining sections develop further material in graph theory and its application; they can be skipped at the first reading.

Section 3.3 introduces the concept of the tree and cotree and discusses the cutset and loopset matrices. Orthogonality relationships of these matrices are demonstrated in Section 3.4, while Section 3.5 shows that independent variables in the network are tree voltages and cotree currents.

So far, the discussion has not distinguished between sources and other elements. The handling of independent sources is described in Section 3.6. The topological formulation of the nodal and loop equations is discussed in Sections 3.7 and 3.8. Finally, the state variable formulation is explained briefly on networks containing R, G, L, and C elements, as well as both types of independent sources.

The topological relations discussed in the first six sections of this chapter are valid for linear as well as nonlinear networks. For this reason, lowercase symbols for voltages and currents will be used. In the remaining sections, the network equations of linear, lumped, time-invariant networks will be considered in the Laplace transform domain and uppercase symbols will be used.

3.1. KVL, KCL, AND ORIENTED GRAPHS

Let us replace each two-terminal element by a line segment called an *edge*. The edges will represent the interconnections or *topology* of the network. Assign distinct numbers to the nodes of the network and the same numbers to the corresponding interconnection points of the edges. The node numbers will be placed within circles to distinguish them from the numbering of the elements in the network or the edges in the graph.

Formulations of network equations are based on the fundamental laws stated first by Kirchhoff. The *Kirchhoff voltage law* (*KVL*) states that the sum of voltage drops around any loop is zero at any instant. The *Kirchhoff current law* (*KCL*) was stated in a restricted form in Chapter 2: the sum of currents flowing away from a node is zero at any moment.

The KCL can be generalized as illustrated with the help of Fig. 3.1.1. Let the network N be partitioned internally into two subnetworks N_1, N_2, joined by

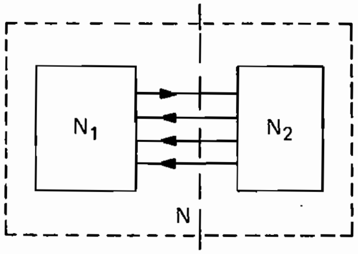


Fig. 3.1.1. Generalization of the KCL.

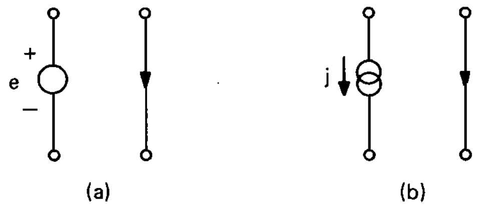


Fig. 3.1.2. Voltage and current sources and their graphs.

a number of interconnections. The sum of all currents in these connections must be zero. The connections separate the two parts or "cut" the network. This will be true for any other set of connections in the network. We may choose only one (any) node as N_1, with the rest of the network being considered in N_2, and obtain the original version of the KCL. We restate the KCL in the following form:

In any cut which separates the network into two parts, the sum of currents in the cut edges is zero.

In the graph representation of a network we replace each two-terminal element by an edge. An orientation will be associated with each edge and we can, for simplicity, assume that for nonsource branches this is the direction of the current flowing through the element. In agreement with Fig. 1.1.1, the orientation will imply that the node from which the current flows is at a higher voltage than the node to which it flows. Since the voltages in the network are not known before the solution, orientations of the graph edges are completely arbitrary for passive elements.

For current sources, the direction of the current is obvious from its symbol. For the voltage source, we will assign the *orientation of the edge as being from* $+$ *to* $-$. The graphs for the sources are given in Fig. 3.1.2. Once all two-terminal elements have been replaced by their oriented edges, there is no reason to distinguish between the sources and other two-terminal elements; all the properties will be derived without reference to the types of elements represented by the edges.

3.2. INCIDENCE MATRIX

Consider the simple network in Fig. 3.2.1(a) and its oriented graph in Fig. 3.2.1(b). Let us write the KCL equations with the edge orientations as indicated. A current flowing *away* from a node will be considered as *positive*. Then

$$1:\ -i_1 + i_4 + i_6 = 0$$

$$2:\ -i_2 - i_4 + i_5 = 0$$

$$3:\ -i_3 - i_5 - i_6 = 0.$$

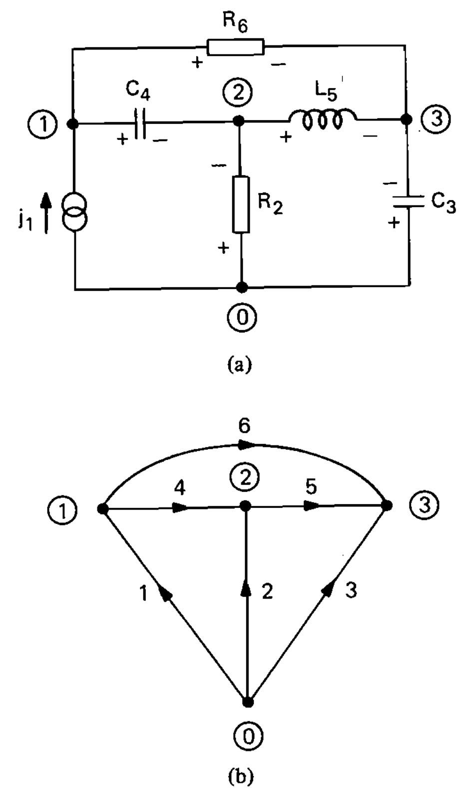


Fig. 3.2.1. A simple network and its oriented graph.

This can be written in matrix form:

$$\mathbf{Ai} = \mathbf{0}. \tag{3.2.1}$$

The matrix **A** is called the *incidence matrix* and, for the example,

$$\mathbf{A} = \begin{array}{c} \\ \text{nodes} \\ \downarrow \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \overset{\text{edges} \longrightarrow}{\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}} \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}. \tag{3.2.2}$$

It has n rows and b columns, n being the number of ungrounded nodes and b

the number of edges in the graph. The rank of this matrix is 3, as the submatrix formed by the first three rows and columns is nonsingular. It can be proven that for a general connected network

$$\text{Rank of } \mathbf{A} = n. \tag{3.2.3}$$

The voltages in the network are also related through the incidence matrix. Denote the voltage of the ith node with respect to the reference node as $v_{n,i}$. Since in our notation the orientation of the edge coincides with the current flow through the element, the node from which the edge leaves will be at a higher potential than the node to which it points. For instance, for the fourth edge in Fig. 3.2.1(b), we will have

$$v_4 = v_{n,1} - v_{n,2}$$

which, in terms of all the nodal variables, is written as

$$v_4 = [1 \quad -1 \quad 0] \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ v_{n,3} \end{bmatrix}.$$

Similarly, for the fifth edge we get the voltage relationship

$$v_5 = [0 \quad 1 \quad -1] \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ v_{n,3} \end{bmatrix}$$

and so on. Writing one such equation for each edge, we arrive at

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ v_{n,3} \end{bmatrix}. \tag{3.2.4}$$

Comparing the matrix in (3.2.4) with the matrix in (3.2.2), we see that it is the transpose of $\mathbf{A}$, $\mathbf{A}'$. The result can be summarized through the matrix equation

$$\mathbf{v} = \mathbf{A}'\mathbf{v}_n. \tag{3.2.5}$$

Equations (3.2.1) and (3.2.5) represent the KCL and KVL relationships for a network, respectively. They will be used repeatedly in subsequent chapters.

3.3. CUTSET AND LOOPSET MATRICES

In order to derive additional properties of oriented graphs, consider Fig. 3.3.1. It shows the same graph discussed in Section 3.2, with some additional features. Recall now the generalized version of the KCL. Three more equations can be written by invoking this form of the law: cut C_4 intersects the edges 6, 4, 2, 3; C_5 intersects the edges 6, 5, 2, 1; and C_6 intersects 1, 4, 5, 3. Selecting the indicated directions with respect to the cut as being positive, we get

$$C_4: \quad i_2 + i_3 + i_4 + i_6 = 0$$

$$C_5: \quad -i_1 - i_2 + i_5 + i_6 = 0$$

$$C_6: \quad i_1 + i_3 - i_4 + i_5 = 0.$$

These equations can be added to the set of equations obtained in Section 3.2. The question arises as to which equations and how many need to be retained to form an independent set. One way of choosing them is by means of the inci-

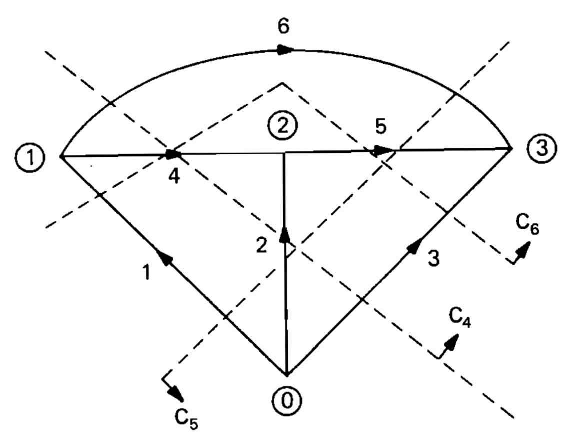


Fig. 3.3.1. Oriented graph with three additional cuts.

dence matrix. It can also be done in a systematic way by defining a tree and a cotree.

A *tree* of a connected graph is a connected subgraph containing all the nodes of the original network but no closed path. A tree of an $n + 1$ node-connected network has n edges.

Any edge in the tree is called a *twig*. The remaining edges, not included in the tree, form a *cotree*, and the edges of a cotree are called *chords*.

A cut with respect to a given tree edge of a graph is a cut going through *only one* twig of the tree and as many chords of the cotree as are necessary to divide the graph into two separate parts. Since there are n twigs of a tree, there will be *n basic cuts*. The set of branches included in any cut is called the *cutset*.

The numbering of the elements and of the nodes is an entirely arbitrary procedure which can in no way alter the behavior of the network. Therefore it is merely a matter of convenience how the various elements are numbered. Many considerations become quite simple if we apply the following rules for the numbering:

1. Assign orientations to the edges.
2. Select a tree.
3. Assign consecutive integers starting from 1 to the twigs and continue numbering the chords.

Figure 3.3.2 indicates one such choice. The edges 1, 2, 3 form the tree of the graph (twigs), and the edges 4, 5, 6 are the chords of the cotree.

To obtain the basic cutsets, we cut one twig and as many chords as are necessary to obtain two disconnected subgraphs. The cuts are indicated in the figure, numbered with subscripts corresponding to the number of the twig they

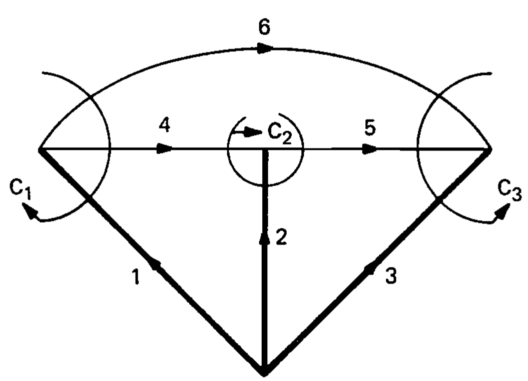


Fig. 3.3.2. Oriented graph with a tree.

cut. Each cut is assigned the same orientation as its twig. This is indicated on the figure by arrows. Then

$$i_1 - i_4 - i_6 = 0$$

$$i_2 + i_4 - i_5 = 0$$

$$i_3 + i_5 + i_6 = 0.$$

In matrix form, this is

$$\mathbf{Qi} = \mathbf{0}. \tag{3.3.1}$$

where

$$\mathbf{Q} = \begin{array}{c} \text{basic} \\ \text{cuts} \\ \downarrow \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \overset{\text{edges} \longrightarrow}{\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}} \atop \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \tag{3.3.2}$$

is called the *basic cutset matrix*.

Before making any statements about the form of the **Q** matrix, let us look at the selection of another tree and the choice of new edge directions for the same network. Consider the graph in Fig. 3.3.3, where the tree is indicated by solid lines and where we followed the numbering suggestions given above. We get

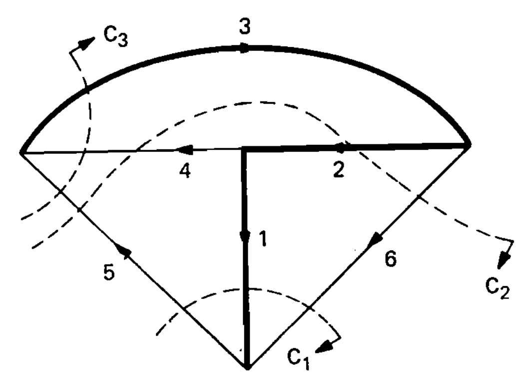


Fig. 3.3.3. Oriented graph with another tree.

$$\mathbf{Q} = \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 2 & 0 & 1 & 0 & -1 & -1 & 1 \\ 3 & 0 & 0 & 1 & -1 & -1 & 0 \end{matrix}.$$

In both cases the *submatrix corresponding to the tree is a unit matrix* and is nonsingular.

The **Q** matrix can be partitioned as follows:

$$\mathbf{Q} = [\mathbf{Q}_t \quad \mathbf{Q}_c] \quad \text{or} \quad [\mathbf{1} \quad \mathbf{Q}_c] \tag{3.3.3}$$

where the subscript t stands for the tree (or twigs), the subscript c for cotree (or chords), and **1** denotes an identity matrix.

The KVL leads to another description of the topological properties of networks. It states that the sum of voltages around any closed path is equal to zero. In our discussion, we will restrict the path to a simple loop, with no crossings and no cases where the path goes two or more times through the same edge or node.

With these restrictions, we can define the following voltage loops in the graph of Fig. 3.3.2:

$$v_1 - v_2 + v_4 = 0$$
$$v_2 - v_3 + v_5 = 0$$
$$v_4 + v_5 - v_6 = 0$$
$$v_1 - v_3 + v_6 = 0$$
$$v_2 - v_3 - v_4 + v_6 = 0$$
$$v_1 - v_2 - v_5 + v_6 = 0$$
$$v_1 - v_3 + v_4 + v_5 = 0$$

and we must decide which of these equations and how many have to be retained to form an independent set. We will apply the concept of the tree and define the loops on the basis of the selected tree.

Consider one chord only and place it in its position in the subgraph formed by the tree. The chord, together with twigs of the tree, will form a loop. In the next step, take the first chord out and place another one in its proper position. This will form another loop. Doing this for all chords, we get a set of loops. Since the network has b edges and $n + 1$ nodes, the tree has n edges and there remain $b - n$ edges to form the chords of the cotree. Therefore, the number of loops formed by the above procedure will be $b - n$.

As before, the sequence in which this is done is arbitrary, as are the orientations in which the loops are closed. Since there is such freedom of choice, we propose a special way of numbering which will considerably simplify the form of the resulting equations:

1. Apply all numbering steps as suggested before.
2. Take the chords in the sequence of their numbering and number the loops, starting from 1.
3. Assume the direction of the loop to be the direction of its chord.

Taking Fig. 3.3.2 with the choice of arrows and numbers as shown, we get three loops:

$$v_1 - v_2 + v_4 = 0$$
$$v_2 - v_3 + v_5 = 0$$
$$v_1 - v_3 + v_6 = 0.$$

In matrix form this is

$$\mathbf{Bv} = \mathbf{0} \tag{3.3.4}$$

where **B** is called the *basic loopset matrix*. Its form is

$$\mathbf{B} = \text{loops}\downarrow \begin{matrix} & \text{edges} \longrightarrow \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & 0 & -1 & 0 & 0 & 1 \end{matrix}. \tag{3.3.5}$$

We observe that the matrix can be partitioned as

$$\mathbf{B} = [\mathbf{B}_t \quad \mathbf{B}_c] \quad \text{or} \quad [\mathbf{B}_t \quad \mathbf{1}]. \tag{3.3.6}$$

Let us see again what happens if another tree and another choice of orientations are used for the same network, provided that the recommended numbering procedure is maintained. Take the tree of Fig. 3.3.3, with the result

$$\mathbf{B} = \text{loops}\downarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \overset{\text{edges} \longrightarrow}{\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}}.$$

Again, $\mathbf{B}$ has a unit submatrix belonging to the chord edges. This submatrix determines the rank of $\mathbf{B}$, which is 3 for our example, and, for a general connected network,

$$\text{Rank of } \mathbf{B} = b - n. \tag{3.3.7}$$

EXAMPLE 3.3.1. Write the $\mathbf{Q}$ and $\mathbf{B}$ matrices for the graph in Fig. 3.3.4. Use the tree formed by the first four edges:

$$\mathbf{Q} = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\mathbf{B} = \left[\begin{array}{cccc|ccc} -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

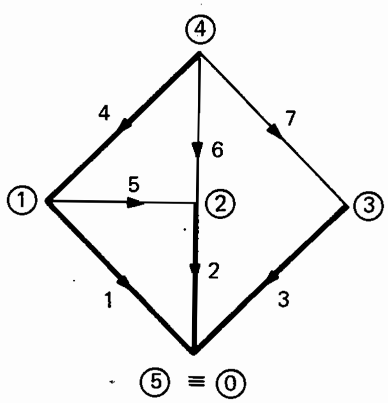


Fig. 3.3.4. A graph with a tree.

3.4. ORTHOGONALITY RELATIONS FOR THE Q AND B MATRICES

The cutset and loopset matrices are not independent, and a fundamental relationship holds. The proof can be found, for instance, in [1].

Arrange the columns of the cutset and loopset matrices in the same sequence of edges; then

$$\mathbf{B}\mathbf{Q}^t = \mathbf{0} \tag{3.4.1}$$

or

$$\mathbf{Q}\mathbf{B}^t = \mathbf{0}. \tag{3.4.2}$$

We will illustrate the validity of the law by considering the matrices (3.3.2) and (3.3.5), written for the graph of Fig. 3.3.2:

$$\mathbf{Q}\mathbf{B}^t = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Important results follow from (3.4.1) or (3.4.2). Using the partitioned forms (3.3.3) and (3.3.6)

$$\mathbf{B}\mathbf{Q}^t = [\mathbf{B}_t \quad \mathbf{1}] \begin{bmatrix} \mathbf{1} \\ \mathbf{Q}_c^t \end{bmatrix} = \mathbf{B}_t + \mathbf{Q}_c^t = \mathbf{0}$$

or

$$\mathbf{B}_t = -\mathbf{Q}_c^t. \tag{3.4.3}$$

Equation (3.4.3) indicates that

$$\mathbf{B} = [-\mathbf{Q}_c^t \quad \mathbf{1}] \tag{3.4.4}$$

and

$$\mathbf{Q} = [\mathbf{1} \quad -\mathbf{B}_t^t] \tag{3.4.5}$$

which means that either the **B** or the **Q** matrix is sufficient for applications. As an exercise, the reader may demonstrate the validity of (3.4.1) and (3.4.2) on the **Q** and **B** matrices of Example 3.3.1.

3.5. INDEPENDENT CURRENTS AND VOLTAGES

The results obtained in the last section will now be used to write the edge voltages and currents in terms of the smaller set of independent voltages and currents. Using the cutset matrix and (3.3.1),

$$\mathbf{Qi} = \mathbf{0}. \tag{3.5.1}$$

Here **i** is the vector of edge currents. Using the partitioning

$$[\mathbf{1} \quad \mathbf{Q}_c] \begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_c \end{bmatrix} = \mathbf{i}_t + \mathbf{Q}_c \mathbf{i}_c = \mathbf{0}$$

or

$$\mathbf{i}_t = -\mathbf{Q}_c \mathbf{i}_c \tag{3.5.2}$$

the twig currents are obtained in terms of chord currents. Since all currents in the network are the union of twig and chord currents, we can write

$$\mathbf{i} = \begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_c \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}_c \\ \mathbf{1} \end{bmatrix} \mathbf{i}_c = \begin{bmatrix} \mathbf{B}_t^t \\ \mathbf{1} \end{bmatrix} \mathbf{i}_c.$$

The last step followed from (3.4.3). However, the last matrix is nothing but the transpose of the matrix **B** (3.3.6); thus,

$$\mathbf{i} = \mathbf{B}^t \mathbf{i}_c. \tag{3.5.3}$$

The chord currents should be considered as independent variables. Once they are known, all the currents in the network are obtained by applying (3.5.3).

Now consider the KVL (3.3.4):

$$\mathbf{Bv} = \mathbf{0} \tag{3.5.4}$$

where **v** is the vector of all edge voltages. Use the partitioning

$$[\mathbf{B}_t \quad \mathbf{1}] \begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_c \end{bmatrix} = \mathbf{B}_t \mathbf{v}_t + \mathbf{v}_c = \mathbf{0}$$

from which

$$\mathbf{v}_c = -\mathbf{B}_t \mathbf{v}_t. \tag{3.5.5}$$

As in the case of the currents, all the edge voltages can be written as

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_c \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t = \begin{bmatrix} \mathbf{1} \\ \mathbf{Q}_c^t \end{bmatrix} \mathbf{v}_t.$$

The last step results from the use of (3.4.3). However, the matrix on the right-hand side is nothing but $\mathbf{Q}^t$. Thus

$$\mathbf{v} = \mathbf{Q}^t \mathbf{v}_t. \tag{3.5.6}$$

We conclude that *twig voltages should be considered as independent voltages.* Once they are known, all edge voltages are obtained by applying (3.5.6).

3.6. INCORPORATING SOURCES INTO GRAPH CONSIDERATIONS

In general, a network can be excited by voltage as well as current sources. The question is: what is an advantageous way of numbering edges in such a case? Equation (3.5.3) indicates that *the chord currents are the independent variables*. The current of an independent current source cannot be a dependent variable. Thus we make sure that *all current sources are placed in the cotree.* Similarly, (3.5.6) indicates that the twig voltages are the independent variables. The voltage of the voltage source cannot be a dependent variable. Thus we must make sure that *all voltage sources are placed in the tree.*

Our preferred numbering system resulted in unit matrices in the place of $\mathbf{Q}_t$ or $\mathbf{B}_c$. In order to preserve this advantage, we will:

1. Represent the voltage sources as $e_1, e_2, \ldots, e_l$ and place them in the tree branches.
2. Complete the tree by taking edges of passive elements and number them by consecutive numbers, starting with 1.

3. Continue numbering of the remaining edges corresponding to the passive elements.
4. Number the current sources as $j_1, j_2, \ldots, j_m$.

The orientations of the sources will be as indicated in Fig. 3.1.2, while the orientations of passive element edges are arbitrary. Whenever the sources are considered explicitly, we will consider the matrices as being *augmented* and use the subscript a.

EXAMPLE 3.6.1. Consider the network shown in Fig. 3.6.1. Write the **Q** matrix by using the numbering on the figure and the suggestions given above.

$$\mathbf{Q}_a = \begin{array}{c} \begin{matrix} e_1 & 1 & 2 & 3 & 4 & 5 & 6 & j_1 \end{matrix} \\ \left[\begin{array}{c:cccccc:c} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array} = [\mathbf{Q}_e \;\; \mathbf{Q} \;\; \mathbf{Q}_j]$$

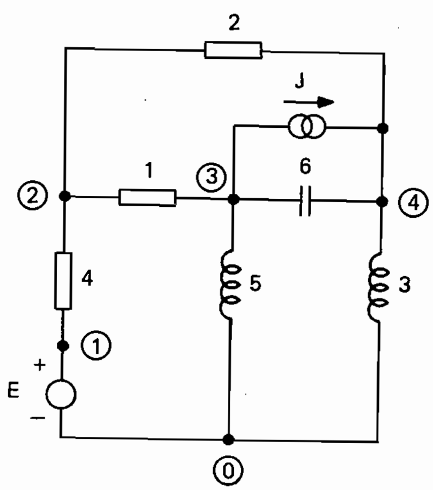


Fig. 3.6.1. Network and its graph with a tree.

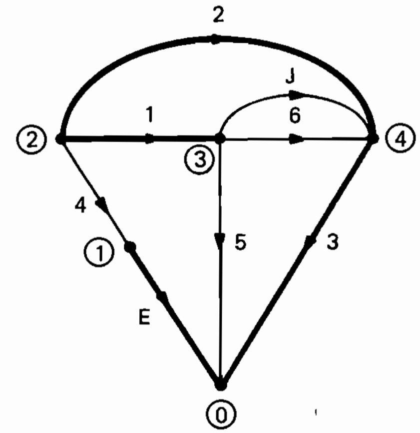


Fig. 3.6.1. *(Continued)*

3.7. TOPOLOGICAL FORMULATION OF NODAL EQUATIONS

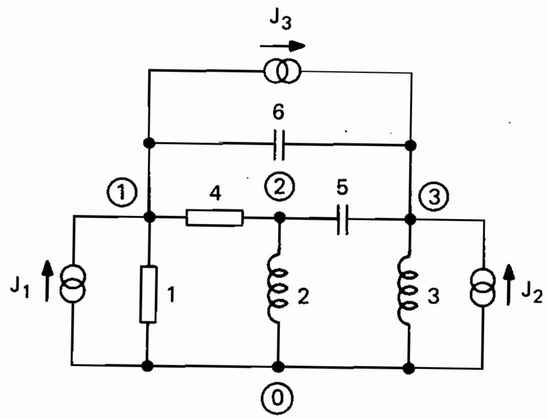
In order to introduce the formulation for the nodal equations, we will assume that *every voltage source has been converted into an equivalent current source* by the Thévenin–Norton transformation.

In the nodal formulation, every passive element is described by the form $Y_b V_b = I_b$. All these expressions can be combined as follows:

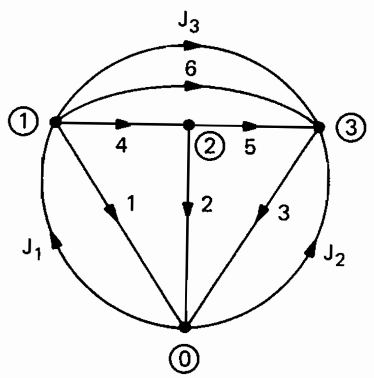
$$\mathbf{Y}_b \mathbf{V}_b = \mathbf{I}_b. \tag{3.7.1}$$

For instance, for the network of Fig. 3.7.1(a), the constitutive equations are

$$\begin{bmatrix} G_1 & & & & & \\ & 1/sL_2 & & & & \\ & & 1/sL_3 & & & \\ & & & G_4 & & \\ & & & & sC_5 & \\ & & & & & sC_6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}.$$



(a)



(b)

Fig. 3.7.1. A network with three current sources and its oriented graph.

The graph for the network is shown in Fig. 3.7.1(b). To write the incidence matrix, consider first the passive elements and then the sources. Thus:

$$\mathbf{A}_a = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & J_1 & J_2 & J_3 \end{matrix} \\ \left[\begin{array}{cccccc:ccc} 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & -1 \end{array}\right] \end{array} = [\mathbf{A} \quad \mathbf{A}_J].$$

Here $\mathbf{A}_a$ is the augmented incidence matrix. The KCL states that

$$\mathbf{A}_a\mathbf{I} = \mathbf{0}. \tag{3.7.2}$$

Partition these equations in the form

$$[\mathbf{A} \quad \mathbf{A}_J]\begin{bmatrix} \mathbf{I}_b \\ \mathbf{J}_b \end{bmatrix} = \mathbf{0} \tag{3.7.3}$$

or

$$\mathbf{A}\mathbf{I}_b = -\mathbf{A}_J\mathbf{J}_b \tag{3.7.4}$$

where $\mathbf{J}_b$ is the vector of known source currents. Substitute (3.7.1) for $\mathbf{I}_b$:

$$\mathbf{A}\mathbf{Y}_b\mathbf{V}_b = -\mathbf{A}_J\mathbf{J}_b. \tag{3.7.5}$$

The KVL is expressed by (3.2.5) for the whole augmented matrix as follows:

$$\mathbf{V} = \mathbf{A}_a^t\mathbf{V}_n. \tag{3.7.6}$$

Again partition the matrix $\mathbf{A}_a^t$:

$$\begin{bmatrix} \mathbf{V}_b \\ \mathbf{V}_J \end{bmatrix} = \begin{bmatrix} \mathbf{A}^t \\ \mathbf{A}_J^t \end{bmatrix}\mathbf{V}_n \tag{3.7.7}$$

where $\mathbf{V}_J$ are the voltages across the sources. Rewrite (3.7.7) as

$$\mathbf{V}_b = \mathbf{A}^t\mathbf{V}_n \tag{3.7.8}$$

$$\mathbf{V}_J = \mathbf{A}_J^t\mathbf{V}_n. \tag{3.7.9}$$

The voltages across the current sources will be determined from (3.7.9) after the node voltages $\mathbf{V}_n$ are found. Insert (3.7.8) into (3.7.5):

$$\mathbf{A}\mathbf{Y}_b\mathbf{A}^t\mathbf{V}_n = -\mathbf{A}_J\mathbf{J}_b \tag{3.7.10}$$

or

$$\mathbf{Y}\mathbf{V}_n = \mathbf{J}_n. \tag{3.7.11}$$

The product

$$\mathbf{Y} = \mathbf{A}\mathbf{Y}_b\mathbf{A}^t \tag{3.7.12}$$

is the *nodal admittance matrix* discussed in Chapter 2. The right-hand side

$$\mathbf{J}_n = -\mathbf{A}_J\mathbf{J}_b \tag{3.7.13}$$

is the *nodal current source* vector of the nodal formulation.

EXAMPLE 3.7.1. Show on the example of Fig. 3.7.1 and on the graph given there that (3.7.10) is equivalent to the nodal formulation of Chapter 2.

The matrices $\mathbf{A}_a$ and $\mathbf{Y}_b$ were already written in this section. The product $\mathbf{A}\mathbf{Y}_b\mathbf{A}^t$ is

$$\mathbf{Y} = \begin{bmatrix} G_1 + G_4 + sC_6 & -G_4 & -sC_6 \\ -G_4 & G_4 + sC_5 + 1/sL_2 & -sC_5 \\ -sC_6 & -sC_5 & sC_5 + sC_6 + 1/sL_3 \end{bmatrix}.$$

For the sources the product is

$$\mathbf{J}_n = -\mathbf{A}_J\mathbf{J}_b = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} J_1 - J_3 \\ 0 \\ J_2 + J_3 \end{bmatrix}.$$

Both results agree with direct analysis.

A similar result can be obtained by using the $\mathbf{Q}$ matrix. The reader may derive the equations as an exercise. A disadvantage of using the basic cutsets is the fact that the resulting admittance matrix will often be dense, in addition to the effort required to obtain $\mathbf{Q}$.

3.8. TOPOLOGICAL FORMULATION OF LOOP EQUATIONS

Formulation of loop equations is simple if we assume that *all current sources have been transformed into voltage sources*. In the loop formulation each passive element is described in the form $Z_b I_b = V_b$, and the expressions can be written compactly as

$$\mathbf{Z}_b\mathbf{I}_b = \mathbf{V}_b. \tag{3.8.1}$$

For instance, for the network of Fig. 3.8.1 the constitutive equations have the form

$$\begin{bmatrix} R_1 & & & & & \\ & 1/sC_2 & & & & \\ & & 1/sC_3 & & & \\ & & & sL_4 & & \\ & & & & sL_5 & \\ & & & & & 1/sC_6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix}.$$

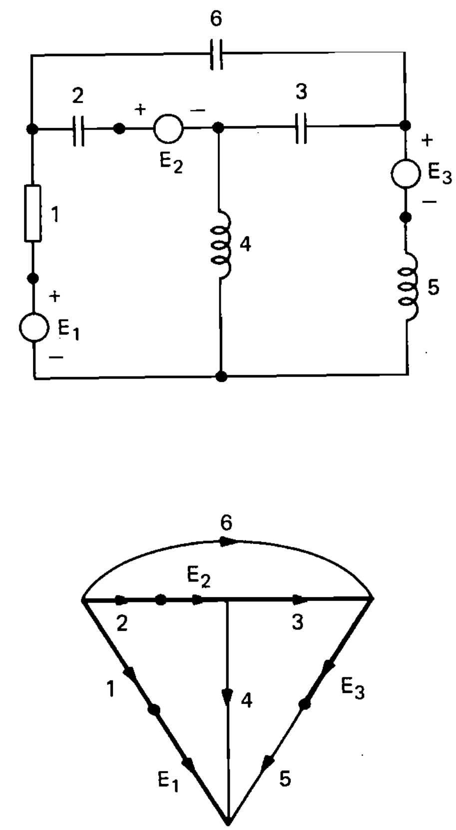


Fig. 3.8.1. A network with three voltage sources and its oriented graph.

The graph of the network is shown as well. The numbering of the branches was done in agreement with Section 3.6. The $\mathbf{B}_a$ matrix is

$$\mathbf{B}_a = \begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} E_1 & E_2 & E_3 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} = [\mathbf{B}_E \quad \mathbf{B}].$$

The subscript a indicates augmentation by sources. The KVL states that

$$\mathbf{B}_a\mathbf{V} = \mathbf{0}. \tag{3.8.2}$$

Partition these equations in the form

$$[\mathbf{B}_E \quad \mathbf{B}]\begin{bmatrix} \mathbf{E}_b \\ \mathbf{V}_b \end{bmatrix} = \mathbf{0} \tag{3.8.3}$$

or

$$\mathbf{B}\mathbf{V}_b = -\mathbf{B}_E\mathbf{E}_b \tag{3.8.4}$$

where $\mathbf{E}_b$ is the vector of known source voltages. Substitute (3.8.1) for $\mathbf{V}_b$:

$$\mathbf{B}\mathbf{Z}_b\mathbf{I}_b = -\mathbf{B}_E\mathbf{E}_b. \tag{3.8.5}$$

The KCL in terms of the **B** matrix was expressed by (3.5.3):

$$\mathbf{I} = \mathbf{B}_a^t\mathbf{I}_c. \tag{3.8.6}$$

Partition this matrix equation as follows:

$$\begin{bmatrix} \mathbf{I}_E \\ \mathbf{I}_b \end{bmatrix} = \begin{bmatrix} \mathbf{B}_E^t \\ \mathbf{B}^t \end{bmatrix}\mathbf{I}_c \tag{3.8.7}$$

where $\mathbf{I}_E$ are the voltage source currents. Rewrite (3.8.7) as

$$\mathbf{I}_E = \mathbf{B}_E^t\mathbf{I}_c \tag{3.8.8}$$

$$\mathbf{I}_b = \mathbf{B}^t\mathbf{I}_c. \tag{3.8.9}$$

Once the currents $\mathbf{I}_c$ are determined, the voltage source currents $\mathbf{I}_E$ will be found from (3.8.8). Insert (3.8.9) into (3.8.5):

$$\mathbf{B}\mathbf{Z}_b\mathbf{B}^t\mathbf{I}_c = -\mathbf{B}_E\mathbf{E}_b \tag{3.8.10}$$

or

$$\mathbf{Z}\mathbf{I}_c = \mathbf{E}_l. \tag{3.8.11}$$

The product

$$\mathbf{Z} = \mathbf{B}\mathbf{Z}_b\mathbf{B}^t \tag{3.8.12}$$

is the *impedance matrix* in the loop formulation and

$$\mathbf{E}_l = -\mathbf{B}_E\mathbf{E}_b \tag{3.8.13}$$

represents the *loop voltages* of the equation. Note that this formulation is *not* identical to the mesh formulation introduced in Chapter 2. The loops are formed here by means of chords. Its advantage, compared with the mesh method, is the fact that it is applicable to planar as well as nonplanar networks. Its disadvantage is that the **Z** matrix is often dense.

EXAMPLE 3.8.1. Derive the matrix **Z** and the source vector $\mathbf{E}_l$ for the network of Fig. 3.8.1. Use the graph and tree given in the figure.

The product $\mathbf{B}\mathbf{Z}_b\mathbf{B}^t$ is

$$\mathbf{Z} = \begin{bmatrix} R_1 + 1/sC_2 + sL_4 & R_1 + 1/sC_2 & -1/sC_2 \\ R_1 + 1/sC_2 & R_1 + 1/sC_2 + 1/sC_3 + sL_5 & -1/sC_2 - 1/sC_3 \\ -1/sC_2 & -1/sC_2 - 1/sC_3 & 1/sC_2 + 1/sC_3 + 1/sC_6 \end{bmatrix}.$$

The source vector is

$$\mathbf{E}_l = -\mathbf{B}_E\mathbf{E}_b = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} E_1 - E_2 \\ E_1 - E_2 - E_3 \\ E_2 \end{bmatrix}.$$

3.9. STATE VARIABLE FORMULATION

In the state variable formulation, all algebraic equations are eliminated and the network is described by a system of first-order differential equations. The formulation is used extensively in theoretical studies, as many theorems from mathematics are directly applicable. In the 1960s, the formulation had some computational justification as well, because good computer codes for solving sets of first-order differential equations were becoming available. The disadvantage of this formulation lies in the complicated steps required to eliminate the algebraic equations and in the fact that the matrices involved are often dense. Algorithms that simultaneously handle sets of differential *and* algebraic equations are now available. Since the disadvantages of the state variable formulation far outweigh its advantages, it is no longer used in computer applications. For this reason, we will restrict our discussion to simple R, L, C, and G elements and independent sources.

The *normal form* of the state equations in the Laplace transform domain is

$$\begin{bmatrix} sX_1 \\ sX_2 \\ \vdots \\ sX_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_m \end{bmatrix}. \tag{3.9.1}$$

or

$$s\mathbf{X} = \mathbf{AX} + \mathbf{BW}. \tag{3.9.2}$$

The standard notation for state variable matrices is used in (3.9.2), where **A** and **B** are *not* the incidence and loopset matrices. The size of **A** is $n \times n$ (n being the number of states), the size of **B** is $n \times m$, and **W** is the vector of independent sources. The normal form of the state equations is usually accompanied by a set of output equations:

$$\mathbf{Y} = \mathbf{CX} + \mathbf{DW} \tag{3.9.3}$$

Y being a vector of length k if there are k outputs. **C** has the dimension $k \times n$ and **D** is $k \times m$. Very often at an intermediate stage of the formulation we get an equation of the form

$$s\mathbf{M}\hat{\mathbf{X}} = \hat{\mathbf{A}}\hat{\mathbf{X}} + \hat{\mathbf{B}}\hat{\mathbf{W}} \tag{3.9.4}$$

where **M** may be a singular matrix. In this section we will indicate the methods for obtaining (3.9.4). Further processing is required to obtain (3.9.2).

Let the network be composed of resistors, capacitors, and inductors. We know that the voltage across an inductor is given by $V = sLI$, whereas the current through the capacitor is given by $I = sCV$. Since (3.9.2) has $s\mathbf{X}$ on the left side, we wish to retain the voltages across the capacitors and the currents through inductors as elements of the vector $s\mathbf{X}$.

It was indicated in Section 3.5 that the twig voltages and chord currents are independent variables. Thus selecting a tree containing all capacitors while all inductors are placed in the cotree solves our problem if such a choice is physically possible.

Further, the voltage of an independent voltage source is specified and *must* be incorporated into the tree. Similarly, the current of an independent current source is given, and it *must* be taken into the cotree. In the selection of a tree (called a *normal* tree), it is best to proceed with the numbering as follows:

1. Take all independent voltage sources into the tree.
2. Take as many capacitors as possible into the tree.
3. Continue taking as many resistors or conductors into the tree as possible.
4. Complete the tree by taking as many inductors as needed.
5. Number the capacitors in the cotree.
6. Number the resistors or conductors in the cotree.
7. Number the inductors in the cotree.
8. Number the independent current sources in the cotree.

To solve the problem, write the **Q** matrix for the network graph. It will have the form $[\mathbf{1}\ \ \mathbf{Q}_c]$. The KCL will have the form (3.5.2), the KVL the form (3.5.5). Writing them in one matrix equation and using (3.4.3):

$$\begin{bmatrix} \mathbf{I}_t \\ \mathbf{V}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{Q}_c \\ \mathbf{Q}_c^t & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_t \\ \mathbf{I}_c \end{bmatrix}. \tag{3.9.5}$$

This is the desired arrangement of variables. On the right side we have the independent twig voltages and chord currents; the dependent variables are on the left side. Equation (3.9.5) is the basis for the state variable formulation.

In the next step, constitutive equations are inserted into (3.9.5) and the matrix equation is rewritten as a set of equations. Finally, only the twig capacitor voltages and chord inductor currents are retained, the other equations being eliminated. This provides a set of equations which can be written in the form (3.9.4).

The steps of the state variable formulation will be explained by means of an example.

EXAMPLE 3.9.1. Find the state variable formulation for the network shown in Fig. 3.9.1. Use the tree indicated in the figure by thicker lines.

The state variables are known; they are the twig capacitor voltages and the chord inductor current, V_{C_2}, V_{C_3}, I_{L_6}. The $\mathbf{Q}$ matrix is

$$\mathbf{Q} = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right] = [\mathbf{1} \quad \mathbf{Q}_c].$$

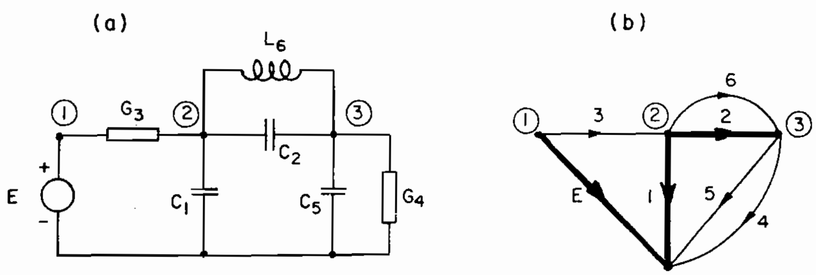


Fig. 3.9.1. Example of state variable formulation.

Equation (3.9.5) becomes

$$\begin{bmatrix} I_E \\ I_1 \\ I_2 \\ \hline V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} V_E \\ V_1 \\ V_2 \\ \hline I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}.$$

This matrix equation expresses all mutual relations between the currents and voltages. The coupling is provided by the branch constitutive equations, which are as follows:

$$\begin{aligned} V_E &= E & I_4 &= G_4 V_4 \\ I_1 &= sC_1 V_1 & I_5 &= sC_5 V_5 \\ I_2 &= sC_2 V_2 & V_6 &= sL_6 I_6. \\ I_3 &= G_3 V_3 \end{aligned}$$

Inserting the constitutive equations into the matrix equation above, we get

$$\begin{bmatrix} I_E \\ sC_1 V_1 \\ sC_2 V_2 \\ V_3 \\ V_4 \\ V_5 \\ sL_6 I_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ V_1 \\ V_2 \\ G_3 V_3 \\ G_4 V_4 \\ sC_5 V_5 \\ I_6 \end{bmatrix}.$$

For hand calculations, the matrix equation is rewritten as a set of equations:

$$I_E = -G_3 V_3$$
$$sC_1 V_1 = G_3 V_3 - G_4 V_4 - sC_5 V_5$$
$$sC_2 V_2 = G_4 V_4 + sC_5 V_5 - I_6$$
$$V_3 = E - V_1$$
$$V_4 = V_1 - V_2$$
$$V_5 = V_1 - V_2$$
$$sL_6 I_6 = V_2.$$

Now eliminate all variables except V_1, V_2, and I_6. The result, in the form of (3.9.4), is

$$s\begin{bmatrix} C_1 + C_5 & -C_5 & 0 \\ -C_5 & C_2 + C_5 & 0 \\ 0 & 0 & L_6 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ I_6 \end{bmatrix}$$
$$= \begin{bmatrix} -(G_3 + G_4) & G_4 & 0 \\ G_4 & -G_4 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ I_6 \end{bmatrix} + \begin{bmatrix} G_3 \\ 0 \\ 0 \end{bmatrix} E.$$

M is nonsingular and can be inverted to obtain the normal form.

PROBLEMS

P.3.1. Draw the oriented graphs for the networks shown in Fig. P.3.1. Use the convention of Fig. 3.1.2 for the sources and assign arbitrary orientations to the edges of all other elements. Write the incidence matrices **A**.

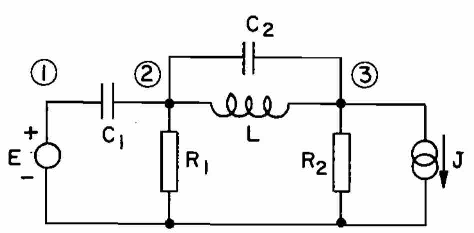


Fig. P.3.1.

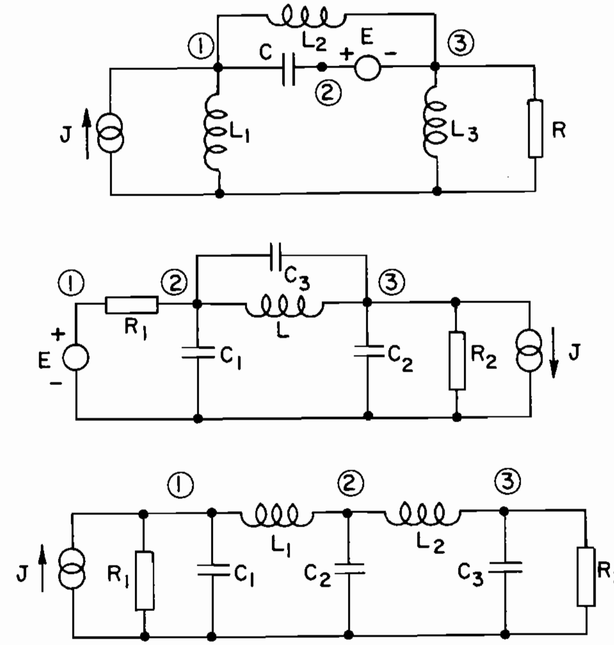


Fig. P.3.1. *(Continued)*

P.3.2. For the networks shown in Fig. P.3.1, write the **Q** and **B** matrices by using the graphs from Problem P.3.1. Select the trees.

P.3.3. Show the validity of the orthogonality relations by using the **Q** and **B** matrices derived in Problem P.3.2.

P.3.4. Transform the voltage sources in Fig. P.3.1 into equivalent current sources, draw appropriate graphs, and apply the topological formulation of nodal equations (Section 3.7) to these networks.

P.3.5. Transform the current sources in Fig. P.3.1 into equivalent voltage sources, draw appropriate graphs, and apply the topological formulation of loop equations (Section 3.8) to these networks.

P.3.6. Apply the state variable formulation to the networks in Fig. P.3.1 by using the **Q** matrices obtained in Problem P.3.2.

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4

General Formulation Methods

The formulation methods introduced in Chapter 2 are quite efficient and have been used successfully in many applications, but they cannot handle all ideal elements. To avoid the restrictions, general formulation methods are introduced in this chapter. In Section 4.1, the tableau formulation [1] is discussed. Here all branch currents, all branch voltages, and all nodal voltages are retained as unknown variables of the problem. Thus the formulation is most general (everything is available after the solution) but leads to large system matrices.

Section 4.2 indicates that blocks of variables can be eliminated and, under special circumstances, this naturally leads to the nodal formulation. However, if we wish to retain the ability to handle all types of network elements, complete block elimination is not possible and the modified nodal formulation [2] must be used. This can be done using graphs, as discussed in Section 4.3, or without graphs, as shown in Section 4.4.

The modified nodal formulations given in Sections 4.3 and 4.4 are efficient but still retain many redundant variables. It is demonstrated in Section 4.5 that active networks can be analyzed extremely efficiently if we follow a set of special rules. The rules given there cannot be easily used for computer solutions, and a systematic method must be found. The basis for eliminating redundant variables is the use of separate voltage and current graphs, discussed in Section 4.6, where they are applied to the tableau formulation. The graphs are a representation of the interconnections and, as such, can be replaced by tables which can be used for automated formulation. Such tabular representation is given in Section 4.7. With this background, the two-graph modified nodal formulation is developed in Section 4.8. Finally, Section 4.9 compares the various formulations introduced in this chapter, and Section 4.10 gives an example.